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1. Introduction

Generalising the well-known result that the wave equations of the electron and the meson can both be put in the linear form

$$\partial_\mu \beta_\mu \psi + \chi\psi = 0 \qquad (\mu = 1, 2, 3, 4) \tag{1}$$

it has been shown elsewhere[6,7,8] that, under suitable assumptions, (1) can also be taken as the wave equation for a particle of arbitrary spin. This is shown possible by postulating the relativistic invariance of (1), and that the spin operator $t_{\mu\nu} = is_{\mu\nu}$ satisfies in every case the equation

$$t_{\mu\nu} = (\beta_\mu \beta_\nu - \beta_\nu \beta_\mu) \equiv (\beta_\mu, \beta_\nu). \tag{2}$$

These assumptions with the further one that the eigen-values of the spin operator for a particle of spin r are $r, r-1, \ldots -r+1, -r$ enable the setting up of a hypercomplex algebra generated by the symbols β_μ and the unit element, reducing in the particular cases of $r = \frac{1}{2}$, and $r = 1$ to the Dirac and meson algebras respectively. The algebra for the case $r = \frac{3}{2}$ can also be dealt with in a simple manner as shown in detail elsewhere.[8]

The particular case of the algebra of meson matrices has been generalised by Schrödinger[9] and Kemmer[3]‡ for the case where more than four elements β_μ exist. The former has considered the case of five elements, and the latter the case of an arbitrary number s of elements. The object of the present paper is to show that this generalisation from four to s elements β_μ is also possible in the case of the algebra associated with a particle of arbitrary spin r or $r + \frac{1}{2}$.

The method adopted in showing this is based on the result deduced by Bhabha[1,2] as a consequence of (2) that the problem of finding irreducible representations of the β-algebra is the same as that of finding all irreducible representations of the Lorentz group in five dimensions (see also Lubański[5]). In fact, this result follows from the relations

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‡ This reference will be denoted hereafter as K.

$$(\beta_\mu, t_{\mu\nu}) = \beta_\nu; \; (t_{\mu\nu}, \beta_\nu) = \beta_\mu \tag{3}$$

which are immediate consequences of (2), and the relativistic invariance of (1). Further, as is well known, the above problem concerning the Lorentz group in five dimensions is identical with the corresponding problem for the real orthogonal group in five dimensions except for the formal process of assigning imaginary values to the parameters involved. It is obvious that, if the relations (2) and (3) also hold when there are s elements β_μ, the representations of the corresponding β-algebra would be the same as those of the real orthogonal group of $s + 1$ dimensions.

Employing this method, we use the well-known results relating to the real orthogonal group of n dimensions ($n = 2k + 1$ or $2k$) to derive the number and order of the several irreducible representations of the generalised s-element β-algebra corresponding to spin r and $r + \frac{1}{2}$ (r being a positive integer). In particular, for $r = 1$ we get an alternative derivation of Kemmer's results for the meson algebra, and equally simple results for the algebra of spin $\frac{3}{2}$.

2. METHOD OF DERIVING REPRESENTATIONS

An alternative formulation of the theory of the linear wave equation (1) has been given by Kramers, Belinfante and Lubański[4] in terms of quantities known as *undors* which transform like products of several Dirac spinors. On this formulation, the wave function in (1), for the general case of spin r, would be an undor of rank $2r$ ($=$ N) and has 4^{N} components. The matrices β_μ are given by

$$\beta_\mu = \sum_{j=1}^{\mathrm{N}} \gamma_{(j)}{}^{\mu}$$

and,

$$\underset{(j)}{\gamma^\mu} = \underset{(1)}{\mathrm{I}} \,\mathrm{X} \ldots \mathrm{X}\, \underset{(j)}{\gamma^\mu}\, \mathrm{X} \ldots \mathrm{X}\, \underset{(\mathrm{N})}{\mathrm{I}}$$

I being the unit matrix, X the direct product, $\gamma_{(j)}{}^{\mu}$ Dirac matrices which commute for different values of j. This representation of the β_μ's is identical with the general spinor representation discovered by Cartan for the orthogonal group and investigated by Brauer and Weyl in detail (see Weyl[11]). As shown by them, every representation of the n-dimensional orthogonal group is contained in the Kronecker product of a certain basic representation with itself taken a sufficient number of times.

Accordingly we assume that the representations of the algebra generated by the β_μ's are contained in the direct product

$$\Delta \times \ldots (2\,r \text{ or } 2\,r+1 \text{ times}) \ldots \times \Delta \tag{4}$$

according as the spin is r or $r+\frac{1}{2}$, where Δ is the basic spin representation of the rotation group in n dimensions (see Murnaghan*[10]). Any general representation of the orthogonal group specified by the k parameters

$$\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_k \geqslant 0 \qquad (n = 2\,k+1, \text{ or } 2\,k)$$

is denoted by $\Gamma_{(\lambda)}$, and the basic representation would, in this notation, be $\Gamma_{(1/2)}{}^{k}$. The successive direct products of Δ by itself can be analysed according to the rule [M., p. 313 (10·21)]

$$\Delta \times \Gamma_{(\lambda)} = \Sigma\, \Gamma_{(p+\frac{1}{2})} \tag{5}$$

the summation on the right being over the 2^k sets (p) which are obtained from the set (λ) by either replacing λ_j by $\lambda_j - 1$ or by leaving it unaltered (it being understood that any set (p) for which the normal non-increasing order is not preserved or which contains a negative number is to be dropped).

The order of any representation $\Gamma_{(\lambda)}$ is given by M. (10·22) and (10·27) or by the equivalent formulæ [M. (9·46) and (9·47), p. 260]

$$d_{(\lambda)} = \frac{2^k}{(2\,k-1)\,!\ldots 3\,!\,!\,!}\; l_1' \ldots l_k' \left\{ \prod_{\substack{(p<q)\\1}}^{k} (l_p'^2 - l_q'^2) \right\} \tag{6, a}$$

for the case $n = 2\,k+1$, and

$$\left.\begin{aligned} d_{(\lambda)} &= \frac{2^k}{(2\,k-2)\,!\ldots 4\,!\,2\,!} \prod_{\substack{(p<q)\\1}}^{k} (l_p^2 - l_q^2);\ l_k \neq 0 \\ &= \frac{2^{k-1}}{(2\,k-2)\,!\ldots 4\,!\,2\,!} \prod_{\substack{(p<q)\\1}}^{k} (l_p^2 - l_q^2);\ l_k = 0 \end{aligned}\right\} \tag{6, b}$$

for the case $n = 2\,k$,

where $\qquad l_1 = \lambda_1 + (k-1),\ l_2 = \lambda_2 + (k-2), \ldots, l_k = \lambda_k \qquad (7)$

and $\qquad l_p' = l_p + \frac{1}{2}$

If we consider the algebra of the s elements β_μ, the order of the corresponding orthogonal group is $s+1$, so that we write

$$\left.\begin{aligned} & n = s+1 \\ \text{and for } s \text{ even,}\quad & n = 2\,k+1 \text{ or, } k = \tfrac{1}{2}s \\ \text{for } s \text{ odd,}\quad & n = 2\,k \text{ or, } k = \tfrac{1}{2}(s+1) \end{aligned}\right\} \tag{8}$$

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* This reference will be denoted hereafter as M.

3. MESON ALGEBRA AND ALGEBRA FOR SPIN 3/2

Let us first consider the meson algebra. The representations of this algebra are contained in the direct product $\Delta \times \Delta$ whose analysis is explicitly given in M. (10·14), p. 309 as

$$\Delta \times \Delta = \left\{ \begin{array}{c} \Gamma_0 + \ldots + \Gamma_k \\ + \Gamma_0^* + \ldots + \Gamma_k^* \end{array} \right\}; \; n = 2k+1 \qquad (9, a)$$

$$\Delta \times \Delta = \left\{ \begin{array}{c} \Gamma_0 + \ldots + \Gamma_{k-1} \\ + \Gamma_0^* + \ldots + \Gamma_{k-1}^* \end{array} \right\} + \Gamma_k (= \Gamma_k' + \Gamma_k''); \; n = 2k \qquad (9, b)$$

Since Γ and Γ^* coincide over the rotation group, we have $(k+1)$ or $(k+2)$ representations according as $n = 2k+1$ or $2k$. The additional representation in the latter case is due to the fact that for $n = 2k$, the two-valued representation Γ_k is the sum of two irreducible inequivalent representations Γ_k' and Γ_k'' of equal orders. In fact, these are the *twin representations* of Kemmer for the case where the number of elements s is odd (K, p. 191). Thus it follows from (8) that the number of representations is $\frac{1}{2}s + 1$ or $\frac{1}{2}(s+5)$ according as s is even or odd [K., p. 192, (8)]. If we discard Γ_0 as belonging to the ablgera corresponding to the lower spin 0, the number of representations would be $\frac{1}{2}s$ or $\frac{1}{2}(s+3)$ according as s is even or odd.

To determine the order d_i of the representation Γ_i in (9, *a*) or (9, *b*), we write the set $(\lambda) = (\lambda_1, \ldots, \lambda_k)$ as $(\lambda_1 = \ldots = \lambda_i = 1;\; \lambda_{i+1} = \ldots = \lambda_k = 0)$, and use (6, *a*) or the second equation of (6, *b*) respectively (since $l_k = 0$). In evaluating the expressions $\Pi(l_p^2 - l_q^2)$ and $\Pi(l_p'^2 - l_q'^2)$ appearing in (6), it introduces simplification if we notice that in the sequence $(l_1, l_2, \ldots, l_k)$ or $(l_1', l_2', \ldots, l_k')$ there is a gap between l_i and l_{i+1}, supply this missing term, and compensate by dividing by the product of the additional squared differences. This procedure leads in the case (6, *a*) to

$$d_i = \frac{2^k}{(2k-1)! \ldots 3!\,1!} \cdot \frac{(2k+1)(2k-1)\ldots 1}{2^k(2k-2i+1)} \cdot (2k)!\,(2k-2)! \ldots 2!$$
$$\div \{i!\,(2k-2i+2)\ldots(2k-i+1)\}\,\{(k-i)!\,(k-i+1)\ldots(2k-2i)\}$$
$$= \frac{(2k-1)!}{i!\,(2k-i+1)!}$$

i.e., $$d_i^{(s)} = \binom{2k+1}{i} = \binom{s+1}{i} \qquad (10)$$

The same result is obtained in the case $n = 2k$ when we use the second equation of (6, *b*). Putting $i = k$, gives, in this case, the order of

Γ_k as $\binom{2k}{k}$, so that each of the twin representations has the order

$$\tfrac{1}{2}\binom{2k}{k} = \tfrac{1}{2}\binom{n}{\frac{1}{2}n} = \tfrac{1}{2}\binom{s+1}{\frac{1}{2}s+\frac{1}{2}}$$

The relation

$$\binom{2k+1}{i} = \binom{2k}{i} + \binom{2k}{i-1}$$

shows that two representations of $(s-1)$ generate one of s. Also, the relation

$$\binom{2k-1}{k-1} = \tfrac{1}{2}\binom{2k}{k}$$

shows that the order of each of the twin representations for odd s is equal to that of the highest representation of $(s-1)$. These results are in accord with those contained in Kemmer's paper (see K, p. 194).

We shall next consider the generalised algebra of spin 3/2 with s symbols β_μ. The representations are given by $\Delta \times \Delta \times \Delta$, and using (9) and writing $\Delta \times \Delta = \sum_0^k \Gamma_i$, we have (for $n = 2k+1$)

$$\Delta \times \Delta \times \Delta = \sum_{i=0}^{k} \Delta \times \Gamma_i = \sum_{i=0}^{k} \Delta \times \Gamma_{(1)^i\,(0)^{k-1}}$$

Analysis by rule (5) leads to

$$\Delta \times \Delta \times \Delta = \sum_{i=0}^{k} (k-i+1)\,\Gamma_{(\frac{3}{2})^i\,(\frac{1}{2})^{k-i}} \tag{11}$$

Thus for even s, there are $(k+1)$ representations given by $\Gamma_{(\frac{3}{2})^i\,(\frac{1}{2})^{k-i}}$ $(i = 0, \ldots, k)$, *i.e.*, $\frac{1}{2}s + 1$ just as in the meson algebra. For the case of odd s the number of representations is again $k + 2 = \frac{1}{2}(s+5)$ as in the meson case since the representation $\Gamma_{(\frac{3}{2})^k}$ is the sum of two twin representations $\Gamma'_{(\frac{3}{2})^k}$ and $\Gamma''_{(\frac{3}{2})^k}$. Discarding in this case $\Gamma_{(\frac{1}{2})^k}$ as belonging to the lower spin $\frac{1}{2}$, *i.e.*, to the s-element Dirac algebra, the number would be $\frac{1}{2}s$ or $\frac{1}{2}(s+3)$.

To determine the order of the representation $\Gamma_{(\frac{3}{2})^i\,(\frac{1}{2})^{k-i}}$, we write $\lambda_1 = \lambda_2 = \ldots = \lambda_i = 3/2$; $\lambda_{i+1} = \ldots = \lambda_k = \frac{1}{2}$, and use (6).

(i) s even, n odd $= 2k+1$; (6, *a*) gives

$$d_i = \frac{2^k\,(2k+1)\,!}{i\,!\,(2k-i+1)\,!}\binom{2k-2i+2}{2k-i+2}$$

i.e, $$d_i = 2^k\left\{\binom{2k+1}{i} - \binom{2k+1}{i-1}\right\} = 2^{\frac{1}{2}s}\left\{\binom{s+1}{i} - \binom{s+1}{i-1}\right\} \tag{12, a}$$

(ii) s odd, n even $= 2\,k$; the first equation of (6, b) gives (since $l_k \neq 0$)

$$d_i = \frac{2^k\,(2\,k)\,!}{i!\,(2k-i)\,!}\left(\frac{2k-2\,i+1}{2k-i+1}\right)$$
$$= 2^k\left\{\binom{2\,k}{i} - \binom{2\,k}{i-1}\right\} = 2^{\frac{1}{2}(s+1)}\left\{\binom{s+1}{i} - \binom{s+1}{i-1}\right\} \qquad (12,\ b)$$

Equations (12, a) and (12, b) are simple generalisations of (10) to the case of spin $\frac{3}{2}$.

That the order of the twin representations for odd s is equal to that of the highest representation for $(s-1)$ also follows immediately from the identity

$$2^{k-1}\left\{\binom{2\,k-1}{k-1} - \binom{2\,k-1}{k-2}\right\} = \tfrac{1}{2}\cdot 2^k\left\{\binom{2\,k}{k} - \binom{2\,k}{k-1}\right\}$$

which can be easily verified.

For the special case of $s = 4$, *i.e.*, the usual case of four β_μ's the equation (12, a) gives the orders of the three representations (corresponding to $i = (0, 1, 2)$ as 4, 16, 20 (see reference 8), and (11) gives the analysis of the triple product of Δ into these components. Since in this case the wave function in (1) is an undor of rank 3 with $4^3 = 64$ components, equation (11) can be expressed by saying that the undor representation for spin $\frac{3}{2}$ contains the 20th order representation once, the 16th order twice, and the Dirac representation thrice, giving the check by degrees,

$$64 = 20 + (2 \times 16) + (3 \times 4).$$

For the general case of s elements also we can obtain the correct check by degrees using (11). Writing $\Delta = \Gamma_{(\frac{1}{2})^k}$, (6) gives the order of this basic representation as 2^k (both for n odd or even). (11) shows that $\Gamma_{(\frac{3}{2})^i\,(\frac{1}{2})^{k-i}}$ is contained $(k-i+1)$ times in $\Delta \mathrm{X}\, \Delta \mathrm{X} \Delta$, and considering $n = 2\,k+1$, (12, a) gives the sum of the orders of the $(k+1)$ representations (including the Dirac representations) as

$$\sum_{i=0}^{k} (k-i+1)\cdot 2^k\left\{\binom{2\,k+1}{i} - \binom{2\,k+1}{i-1}\right\}$$
$$= 2^k\,(n\,\mathrm{C}_0 + n\,\mathrm{C}_1 + \ldots . + n\,\mathrm{C}_k)$$
$$= \tfrac{1}{2}\cdot 2^k\,(n\,\mathrm{C}_0 + n\,\mathrm{C}_1 + \ldots . + n\,\mathrm{C}_n) \text{ since } n \text{ is odd,}$$
$$= \tfrac{1}{2}\cdot 2^k\cdot 2^{2k+1} = 2^{3k}$$

which is equal to the order of the direct product $\Delta \mathrm{X}\ \Delta \mathrm{X}\ \Delta$ since Δ is of order 2^k.

For the case $n = 2\,k$, a correct check by degrees using (12, b) requires that the representations Γ_i $(i = 0, \ldots ., \overline{k-1})$ should each appear twice as many times as they do in the case $n = 2k + 1$. The same situation also holds in the meson case as can be seen from the table given by Kemmer (K, p. 195).

4. ALGEBRA FOR SPINS r AND $r + \frac{1}{2}$

The derivation of the representations for $r = \frac{3}{2}$ from those of $r = 1$ suggests that a typical representation in the general case of r and $r + \frac{1}{2}$ would be of the forms

$$\Gamma_{(r)^{k-i_1}\,(r--1)^{i_1-i_2}\ldots.(1)^{i_{r-1}-i_r}\,(0)^{i_r}} \qquad \text{(spin } r\text{)} \qquad (13,\ a)$$

and,

$$\Gamma_{(r+\frac{1}{2})^{k-i_1}\,(r-\frac{1}{2})^{i_1-i_2}\ldots.(\frac{3}{2})^{i_{r-1}-i_r}\,(\frac{1}{2})^{i_r}} \qquad \text{(spin } r+\tfrac{1}{2}\text{)} \qquad (13,\ b)$$

labelled by r indices $i_1, i_2, \ldots. i_r$ $(i_1 \geqslant i_2 \geqslant \ldots. \geqslant i_r \geqslant 0)$ taking the values

$$\left.\begin{array}{l} i_r = 0,\ 1,\ 2, \ldots. i_{r-1} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ i_j = 0,\ \ldots\ldots.. i_{j-1} \\ i_2 = 0,\ \ldots\ldots\ldots. i_1 \\ i_1 = 0, \ldots\ldots(k-1) \end{array}\right\} \qquad (14)$$

The restriction of the values of i_1 in (14) from 0 to $(k - 1)$ instead of 0 to k serves to eliminate those representations which belong to the lower spins $(r - 1)$ and $(r - \frac{1}{2})$ as can be seen from (13, *a*) and (13, *b*) respectively. It is also to be noticed that the order in which the representations are numbered by the suffixes $i_1, i_2, \ldots. i_r$ in (13) is different from the order adopted in the particular case of $r = 1$. In fact, putting $i_2 = i_3 = \ldots. = i_r = 0$, the representation denoted by the single index i_1 would correspond to Γ_{k-i_1} in the notation of the previous article. We shall however denote the representation given by (13) as $\Gamma_{i_1, i_2, \ldots. i_r}$ and derive the total number of these for the spins r and $r + \frac{1}{2}$. It is obvious that the number is the same for both the spins.

This total number is the same as the number of ways of choosing the indices subject to the conditions (14). For a given i_j the index i_{j+1} takes on $(i_j + 1)$ values, *viz.*, $(0, 1, \ldots, i_j)$ while i_1 takes on k values, *viz.*, $(0, 1, \ldots. \overline{k - 1})$. Thus the number required is the number of ways of choosing r things out of $(r + k - 1)$, and hence

$$\text{number of representations} = \binom{k + r - 1}{r} \qquad (15)$$

We shall add a proof by induction. Assuming the truth of (15) for r we obtain the number of representations for $(r + 1)$ with the aid of the typical one (r integral)

$$\Gamma_{(r+1)^{k-i_1}(r)^{i_1-i_2}\ldots(0)^{i_r+1}} \tag{13, c}$$

as $\sum\limits_{i_1=0}^{(k-1)} \binom{i_1+r}{r}$, since in (13, c) the index i_2 assumes values from 0 to i_1. But

$$\sum_{i_1=0}^{k-1} \binom{i_1+r}{r} = \sum_{l=1}^{k} \binom{l+r-1}{r}$$

$$= \sum_{l=1}^{k} \left\{ \binom{l+r}{r+1} - \binom{l+r-1}{r+1} \right\} \text{ using } nC_{r-1} = (n+1)\,C_r - nC_r$$

$$= \binom{k+r}{r+1} - \binom{r}{r+1}$$

$$= \binom{k+r}{r+1}, \text{ since the second term vanishes}$$

which is (15) wth $(r+1)$ in place of r. Also (15) is true for $r=1$, and hence the validity of (15) is proved for general r.

The representation of the highest order is given by putting $i_1 = i_2 = \ldots = i_r = 0$ as $\Gamma_{(r)^k}$ or $\Gamma_{(r+\frac{1}{2})^k}$, and in the case $n = 2k$, this is a two-valued representation which is the sum of two irreducible inequivalent '*twin*' representations of equal orders. Thus we have (both for spins r and $r+\frac{1}{2}$)

(i) for s even, $n = s+1 = 2k+1$,

$$\text{no. of representations} = \binom{k+r-1}{r} = \binom{s+r-1}{r} \tag{15, a}$$

(ii) for s odd, $n = s+1 = 2k$,

$$\text{no. of representations} = \binom{k+r-1}{r} + 1 = \binom{\frac{1}{2}s+r-\frac{1}{2}}{r} + 1 \tag{15, b}$$

In the usual case of $s=4$, *i.e.*, $k=2$, (15, a) gives the number as $\binom{r+1}{r} = (r+1)$. For $r=1$, we get the results already obtained for the meson case.

Next to determine the order of the typical representation $\Gamma_{i_1, i_2, \ldots i_r}$, we consider the cases r and $r+\frac{1}{2}$ separately.

Case (*i*)—*Spin r* (*integral*).—For the representation (13, a), the (λ)-set is given by the scheme

$$(\lambda) = (r, r, \ldots r;\ \overline{r-1}, \ldots, \overline{r-1};\ \ldots;\ 1, \ldots 1;\ 0, \ldots 0) \tag{16·1}$$

the successive integers being repeated $(k-i_1)$, (i_1-i_2), …, i_r times respectively. Therefore the (l)-set is given by the scheme

$$l) = (k+r-1, \ldots, i_1+r;\ i_1+r-2, \ldots, i_2+r-1;\ i_2+r-3, \ldots 1, 0) \tag{17·1}$$

(*a*) $n = 2k + 1$.—We use (6, *a*), and introduce in the sequence l_p' the missing terms $(i_1 + r - \frac{1}{2})$, $(i_2 + r - \frac{3}{2}), \ldots.$ while forming the product $\Pi\,(l_p'^2 - l_q'^2)$, and then divide by the additional factors thus introduced. This gives, after some easy simplification, the order of the representation as

$$d_{i_1\, i_2, \ldots . i_r} = \frac{\prod\limits_{p=1}^{r} (2k + 2r - 2p + 1)\,!\ \ \underset{p=1}{\overset{r}{\Delta}} \{(i_p + r - p + \tfrac{1}{2})^2\}}{\prod\limits_{p=1}^{r} (k - i_p + p - 1)\,!\ \ \prod\limits_{p=1}^{r} (k + i_p + 2r - p)\,!} \tag{18, a}$$

where Δ will be used hereafter to denote the difference product. We can transform (18, *a*) to an elegant determinantal form by writing

$$(i_p + r - p + \tfrac{1}{2}) - (i_q + r - q + \tfrac{1}{2}) = (k - i_q + q - 1) - (k - i_p + p - 1)$$

and,

$$(i_p + r - p + \tfrac{1}{2}) + (i_q + r - q + \tfrac{1}{2}) = (k + i_q + 2r - q) - (k - i_p + p - 1)$$

and reducing it to a form in which there are only products of factorial expressions both in the numerator and denominator. This reduction gives

$$d_{i_1\, i_2 \ldots . i_r} = \begin{vmatrix} \binom{2k + 2r - 1}{k - i_1}, & \binom{2k + 2r - 1}{k - i_2 + 1}, & \ldots \binom{2k + 2r - 1}{k - i_r + r - 1} \\ \binom{2k + 2r - 3}{k - i_1 - 1}, & \binom{2k + 2r - 3}{k - i_2}, & \ldots \binom{2k + 2r - 3}{k - i_r + r - 2} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \binom{2k + 1}{k - i_1 - r + 1}, & \binom{2k + 1}{k - i_2 - r + 2}, & \ldots \binom{2k + 1}{k - i_r} \end{vmatrix} \tag{19, a}$$

which is a determinant of the r-th order. In the abbreviated form $|\,a_{pq}\,|$ of a determinant the element in the p-th row and q-th column of which is a_{pq} we can write

$$d_{i_1 \ldots . i_r} = \left| \binom{2k + 2r - 2p + 1}{k - i_q + q - p} \right| \tag{20, a}$$

(*b*) $n = 2k$.—We use the same (λ) and (l)-sets as in (15, *a*) and (17, *a*), but the second formula of (6, *b*) since $l_k = 0$. This gives

$$d_{i_1\, i_2 \ldots . i_r} = \frac{\prod\limits_{p=1}^{r} (2k + 2r - 2p)\,!\ \ \underset{p=1}{\overset{r}{\Delta}} \{(i_p + r - p)^2\}}{\prod\limits_{p=1}^{r} (k - i_p + p - 1)\,!\ \ \prod\limits_{p=1}^{r} (k + i_p + 2r - p - 1)\,!} \tag{18, b}$$

or in the determinantal form

$$d_{i_1 i_2 \ldots i_r} = \begin{vmatrix} \binom{2k+2r-2}{k-i_1}, & \binom{2k+2r-2}{k-i_2+1}, & \ldots \binom{2k+2r-2}{k-i_r+r-1} \\ \binom{2k+2r-4}{k-i_1-1}, & \binom{2k+2r-4}{k-i_2}, & \ldots \binom{2k+2r-4}{k-i_r+r-2} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \binom{2k}{k-i_1-r+1}, & \binom{2k}{k-i_2-r+2}, & \binom{2k}{k-i_r} \end{vmatrix} \qquad (19,\ b)$$

a determinant again of order r which can be abbreviated as

$$d_{i_1 i_2 \ldots i_r} = \left| \binom{2k+2r-2p}{k-i_q+q-p} \right| \qquad (20,\ b)$$

Case (ii).—Spin $(r+\frac{1}{2})$ *(half-integral)*—For the representation (13, *b*) the (λ)-set is given by

$$(\lambda) = (r+\tfrac{1}{2},\ \ldots,\ r+\tfrac{1}{2};\ r-\tfrac{1}{2},\ \ldots,\ r-\tfrac{1}{2};\ \ldots;\ \tfrac{3}{2},\ \ldots,\ \tfrac{3}{2};\ \tfrac{1}{2},\ \ldots,\ \tfrac{1}{2}) \qquad (16\cdot 2)$$

the successive half-integers $(r+\frac{1}{2})$, $(r-\frac{1}{2})$, ..., $\frac{1}{2}$ being repeated $(k-i_1)$, (i_1-i_2), ..., i_r times respectively. Therefore from (7) the (l)-set is given by

$$(l) = (k+r-\tfrac{1}{2},\ \ldots,\ i_1+r+\tfrac{1}{2};\ i_1+r-\tfrac{3}{2},\ \ldots,\ i_2+r-\tfrac{1}{2};\ i_2+r-\tfrac{5}{2},\ \ldots,\ \tfrac{3}{2},\ \tfrac{1}{2}) \qquad (17\cdot 2)$$

(*a*) $n = 2k+1$.—We use (6, *a*) and proceed as in the case of integral spin to derive

$$d_{i_1 \ldots i_r} = 2^k \frac{\prod_{p=1}^{r} (2k+2r-2p+1)!\ \overset{r}{\underset{p=1}{\Delta}} \{(i_p+r-p+1)^2\} \prod_{p=1}^{r} (2i_p+2r-2p+2)!}{\prod_{p=1}^{r} (k-i_p+p-1)!\ \prod_{p=1}^{r} (k+i_p+2r-p+1)!} \qquad (21,\ a)$$

Comparing this with (18, *a*) we notice that (21, *a*) has an additional factor appearing in the numerator, and in the transformation to the determinantal form we also use

$$(2i_p+2r-2p+2) = (k+i_p+2r-p+1) - (k-i_p+p-1),$$

and this leads from (21, *a*) to the expression

$$d_{i_1 i_2 \ldots i_r} = 2^k \Sigma \pm \left| \binom{2k+2r-2p+1}{k-i_q+q-p} \right| \qquad (22,\ a)$$

The 2^r determinants in the summation in (22, *a*) are derived from (20, *a*) by diminishing the indices $i_1, i_2, \ldots, i_r$ by one, one at a time, two at a a time, and to on, and attaching the ± sign according as the number of indices decreased by one be even or odd respectively. In other words, the several determinants in (22, *a*) are derived from (20, *a*) by diminishing by one the expression $(k - i_q + q - p)$ in one column, two columns, and so on at a time. It might be observed that the summation in (22, *a*) is analogous to that on the right-hand side of (5).

(*b*) $n = 2k$.—The (λ) and (l)-sets are the same as in (16·2) and (17·2) but we use the first formula of (6, *b*) since $l_k \neq 0$. This gives

$$d_{i_1 \ldots i_r} = 2^k \frac{\prod\limits_{p=1}^{r} (2k+2r-2p)! \; \overset{r}{\underset{p=1}{\Delta}} \{(i_p+r-p+\tfrac{1}{2})^2\} \prod\limits_{p=1}^{r} (2i_p+2r-2p+1)!}{\prod\limits_{p=1}^{r} (k-i_p+p-1)! \; \prod\limits_{p=1}^{r} (k+i_p+2r-p)!} \tag{21, b}$$

In terms of determinants this reduces just like (22, *a*) to

$$d_{i_1 i_2 \ldots i_r} = 2^k \, \Sigma \pm \left| \binom{2k + 2r - 2p}{k - i_q + q - p} \right| \tag{22, b}$$

The orders of the several representations given by (20) and (22) can be expressed in terms of s by using (8).

5. Discussion of Results

The representations of the highest order are given by setting $i_1 = i_2 = \ldots = i_r = 0$ in the fromulæ of the previous section. We shall denote these by $d^{(r,s)}_{\max}$, and derive the relation between $d^{(r,s-1)}_{\max}$ and $d^{(r,s)}_{\max}$ when s is odd. From (18, *b*)

$$d^{(r,s)}_{\max} = \frac{\Pi \; (2k + 2r - 2p)\,! \; \{\Delta \; (r - p)^2\}}{\Pi \, (k + p - 1)\,! \; \Pi \, (k + 2r - p - 1)\,!}$$

Again, from (18, *a*), putting $i_p = 0$, and changing k into $(k - 1)$

$$d^{(r,s-1)}_{\max} = \frac{\Pi \, (2k + 2r - 2p - 1)\,! \;\; \Delta \, \{(r - p + \tfrac{1}{2})^2\}}{\Pi \, (k + p - 2)\,! \; \Pi \, (k + 2r - p - 1)\,!}$$

Now,

$$\Delta \, \{(r - p)^2\} = (r - 1)\,! \prod_{1}^{(r-1)} (2r - 2p - 1)\,!$$

and

$$\Delta \, \{(r - p + \tfrac{1}{2})^2\} = \prod_{1}^{(r-1)} (2r - 2p)\,!$$

Hence

$$\frac{d^{(r,s)}_{\max}}{d^{(r,s-1)}_{\max}} = \frac{\Pi \, (2k + 2r - 2p) \cdot (r - 1)\,!}{\Pi \, (k + p - 1) \prod\limits_{1}^{(r-1)} (2r - 2p)} = 2^r \cdot \frac{(r - 1)\,!}{2^{r-1} (r - 1)\,!} = 2$$

Thus,

$$d^{(r,s-1)}_{\max} = \tfrac{1}{2} \, d^{(r,s)}_{\max} \tag{23, a}$$

In general, therefore, the order of each of the twin representations for odd s is equal to that of the highest order reprsenetation for $(s-1)$. In a similar way, using (21, *a*) and (21, *b*) we can show that for $n = 2k$ (s odd)

$$d_{\max}^{(r+\frac{1}{2},\, s-1)} = \tfrac{1}{2}\, d_{\max}^{(r+\frac{1}{2},\, 's)} \tag{23, b}$$

A simpler proof is also possible using (20, *a*) and (20, *b*).

For the usual case of four elements β_μ, *i.e.*, $s = 4$, formulæ (18, *a*) and (21, *a*) give the dimensions of the representations for spins r and $r + \frac{1}{2}$ by putting $k = 2$. Considering (18, *a*) first, let us take the choice of indices given by

$$i_1 = i_2 + \ldots . = i_j = 1\,;\; i_{j+1} = \; \ldots . = i_r = 0 \tag{24}$$

This gives after some simplification

$$d_{(1)^j\,(0)^{r-j}} = \tfrac{1}{6}\,(2r+3)\,(j+1)\,(2r-2j+1)\,(2r-j+2)$$

Changing j into $(r-j)$, this gives

$$d_{(1)^{r-j}\,(0)^j} = \tfrac{1}{6}\,(2r+3)\,(r-j+1)\,(2j+1)\,(r+j+2) \tag{25}$$

We can replace in this r and j by the indices λ_1, λ_2 and the general representation for spin λ_1 characterised by (λ_1, λ_2) where λ_2 takes on values from 0 to λ_1, has therefore the dimension

$$\begin{aligned} d_{(\lambda_1,\, \lambda_2)} &= \tfrac{1}{6}\,(2\,\lambda_1 + 3)\,(\lambda_1 - \lambda_2 + 1)\,(2\,\lambda_2 + 1)\,(\lambda_1 + \lambda_2 + 2) \\ &= \tfrac{2}{3}\,(\lambda_1 + \tfrac{3}{2})\,(\lambda_2 + \tfrac{1}{2})\,(\lambda_1 - \lambda_2 + 1)\,(\lambda_1 + \lambda_2 + 2) \end{aligned} \tag{26}$$

which is identical with the formula in M. p. 314 for the case $n = 5$, $k = 2$. It can be shown that the substitution (24) in (21, *a*) also leads to the same formula (26) with $\lambda_1 = r + \frac{1}{2}$, $\lambda_2 = j$.

6. SUMMARY

It is shown that the results obtained by Schrödinger, and Kemmer for the s-element meson algebra also hold for the generalisation, to s elements, of the algebra corresponding to general spins r and $(r + \frac{1}{2})$ previously set up by the author. The representations of this algebra are obtained by analysing the direct product of the basic representation of the rotation group of $(s + 1)$ dimensions with itself taken $2\,r$ or $2\,r + 1$ times. The total number of representations (excluding those belonging to lower spins) is obtained as $(k + r - 1)\,C_r$ or $(k + r - 1)\,C_r + 1$ for $n = s + 1 = 2k+1$ or $2\,k$ respectively, both for spins r and $(r + \frac{1}{2})$. The order of a general representation specified by r indices $i_1, i_2, \ldots, i_r$ is given in a simple determinantal form in the four cases of $n = 2\,k + 1$ or $2\,k$, and spins r or $(r + \frac{1}{2})$. It is also shown in general that the twin representation for odd s has the same order as that of the highest one for $(s - 1)$ elements.

The method of the paper leads to an alternative derivation of Kemmer's results for the meson case, and to equally simple results for the generalised *s*-element algebra of spin $\frac{3}{2}$.

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